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Noncommutative and ordinary super Yang-Mills on (D$(p-2)$, Dp) bound states

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Abstract: We study properties of (D$(p-2)$, Dp) nonthreshold bound states ($2 \leq p \leq 6$) in the dual gravity description. These bound states can be viewed as Dp-branes with a nonzero NS B-field of rank two. We find that in the decoupling limit, the thermodynamics of the N_p coincident Dp-branes with B-field is the same not only as that of N_p coincident Dp-branes without B-field, but also as that of the N_{p-2} coincident D$(p-2)$-branes with two smeared coordinates and no B-field, for $N_{p-2}/N_p = \tilde{V}_2/[(2\pi)^2\tilde{b}]$ with $\tilde{V}_2$ being the area of the two smeared directions and $\tilde{b}$ a noncommutativity parameter. We also obtain the same relation from the thermodynamics and dynamics by probe methods. This suggests that the noncommutative super Yang-Mills with gauge group U(N_p) in $(p+1)$ dimensions is equivalent to an ordinary one with gauge group U(∞) in $(p-1)$ dimensions in the limit $\tilde{V}_2 \to \infty$. We also find that the free energy of a Dp-brane probe with B-field in the background of Dp-branes with B-field coincides with that of a Dp-brane probe in the background of Dp-branes without B-field.

Keywords: D-branes, String Duality.

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Contents

1. Introduction

Recently there has been much interest in studying the properties of the (D($p-2$), Dp) bound states and their consequences (see for example [1]–[11] and references therein). Such bound states can be viewed as Dp-branes with a nonzero (rank two) Neveu-Schwarz (NS) B-field. At present, it is known that the worldvolume coordinates will become noncommutative if a Dp-brane carries a nonvanishing NS B-field on its worldvolume [12]–[20], and the field theories on the worldvolume of such D-branes are called noncommutative field theories in order to distinguish the ordinary field theories on the worldvolume of D-branes without NS B-field. The gauge theories on the noncommutative spacetimes can naturally be realized in string theories. According to the Maldacena conjecture [21]–[24], the string theories can be used to study the large-N noncommutative field theories in the strong 't Hooft coupling limit.

On the basis of consideration of planar diagrams, Bigatti and Susskind [25] argued that the large-N noncommutative and ordinary gauge field theories are equivalent in the weak coupling limit and noncommutative effects can be seen only in the nonplanar diagrams. Explicit perturbative calculations [26] render evidence to this assertion. On the supergravity side, it has also been found that the thermodynamics of near-extremal Dp-branes with a nonvanishing NS B-field coincides exactly with that of the corresponding Dp-branes without B field [2, 5, 6, 7, 9], which means that in the large N and strong coupling limit, the number of the degrees of freedom of the noncommutative gauge fields remains unchanged despite the noncommutativity of space.

More recently, Lu and Roy [11] have found that in the system of (D$(p-2)$, Dp) bound states ($2 \leq p \leq 6$), the noncommutative effects of gauge fields are actually due to the presence of *infinitely many* D$(p-2)$-branes which play the dominant role over the Dp-branes in the large B-field limit. The Dp-branes with a constant B-field represent dynamically the system of infinitely many D$(p-2)$-branes with two smeared transverse coordinates (additional isometrics) and no B-field in the decoupling limit. With this observation, Lu and Roy further argued that there is an equivalence between the noncommutative super Yang-Mills theory in $(p+1)$ dimensions and an ordinary one with gauge group U(∞) in $(p-1)$ dimensions. For related discussions, see also [27, 28, 4, 29, 30, 31].

In the present paper, we would like to discuss this equivalence from the viewpoint of the thermodynamics of the (D$(p-2)$, Dp) bound states in the dual gravity description with two dimensions compactified on a torus. The case discussed by Lu and Roy [11] corresponds to the infinite volume limit of the torus, and we would like to clarify some subtle questions in this analysis.

In the next section, we study the black (D$(p-2)$, Dp) configuration and some of its basic thermodynamic properties. As mentioned above, it has been noticed that the thermodynamics of the black Dp-branes with B-field is the same as that of Dp-branes without B-field. We show here that the thermodynamics of the Dp-branes with nonzero B-field is also completely the same as that of D$(p-2)$-branes with two smeared coordinates and zero B-field. We obtain a relation (eq. (2.21) below) between the numbers of Dp-branes with B-field and the D$(p-2)$-branes without B-field when this equivalence is valid. Since the worldvolume theory of Dp-branes with B-field is the noncommutative super Yang-Mills with gauge group U(N_p) in $(p+1)$ dimensions with two dimensions compactified on a torus and that of D$(p-2)$-branes with two smeared coordinates and zero B field is an ordinary super Yang-Mills with gauge group U(N_{p-2}) in $(p+1)$ dimensions on the dual torus, as we will see, this implies that these theories are equivalent in the large-N limit. When the volume of the torus is sent to infinity, the latter theory reduces to U(∞) gauge theory in $(p-1)$ dimensions for fixed N_p, in agreement with [11]. This is also in accordance with the proposal that D2-brane is a condensate of D0-branes [27, 28, 4]. Furthermore, we give the relation of Yang-Mills coupling constants between the theories in different dimensions.

Investigating the interactions between a probe and a source is a useful method to see some of the properties of the source. For instance, a scalar field has been used [32, 33, 34] as a probe to find out the noncommutative effects on the absorption by the (D1, D3) bound state. In section 3, we study the descriptions of the (D$(p-2)$, Dp) bound state in terms of the Dp-branes with B-field and in terms of the D$(p-2)$-branes without B-field by analyzing the thermodynamics of two probes, one of which is a bound state of D$(p-2)$- and Dp-branes and the other is a D$(p-2)$-brane. In the decoupling limit, we find a relation (3.19) similar to (2.21). In section 4, we further reveal the equivalence of the descriptions by examining the dynamics of the probes in the bound state backgrounds. We find that non-extremal D-branes can be located at the horizon from the viewpoint of probes. We summarize our results in section 5.

2. The (D$(p-2)$, Dp) bound states and implications of their thermodynamics

The supergravity solutions of the (D$(p-2)$, Dp) bound states ($2 \leq p \leq 6$) in type-II superstring theories have been constructed by many authors in [35, 36, 37, 38, 9]. The supergravity solutions of the (D$(p-2)$, Dp) bound states can be acquired by taking the extremal limit of the corresponding black configurations.

We start with the general solution

$$\begin{aligned} ds^2 &= H^{-1/2}\left[-f dt^2 + dx_1^2 + \cdots + dx_{p-2}^2 + h(dx_{p-1}^2 + dx_p^2)\right] + \\ &\quad + H^{1/2}\left(f^{-1}dr^2 + r^2 d\Omega_{8-p}^2\right), \\ e^{2\phi} &= g^2 H^{\frac{3-p}{2}} h\,, \qquad B_{p-1,p} = \tan\theta H^{-1} h\,, \\ A^p_{012\cdots p} &= g^{-1}(H^{-1}-1) h \cos\theta \coth\alpha\,, \\ A^{p-2}_{012\cdots(p-2)} &= g^{-1}(H^{-1}-1)\sin\theta\coth\alpha\,, \end{aligned} \tag{2.1}$$

and

$$H = 1 + \frac{r_0^{7-p}\sinh^2\alpha}{r^{7-p}}\,, \qquad f = 1 - \left(\frac{r_0}{r}\right)^{7-p}\,, \qquad h^{-1} = \cos^2\theta + H^{-1}\sin^2\theta\,. \tag{2.2}$$

Here g is the string coupling constant, r_0 is the non-extremal Schwarzschild mass parameter and α is the boost parameter. The solution (2.1) interpolates between the black D$(p-2)$-brane solution with two smeared coordinates x_{p-1} and x_p ($\theta = \pi/2$), and the black Dp-brane with zero B-field ($\theta = 0$). Note that a constant part of the B-field can be gauged away so that the constant value for $B_{p-1,p}$ can be changed. The parameter θ characterizes the interpolation. The coordinates x_{p-1} and x_p are relative transverse directions for the D$(p-2)$-branes and parametrize a rectangular 2-torus.

Denote the area of the 2-torus spanned by x_{p-1} and x_p by V_2 and the spatial volume of the D$(p-2)$-brane with worldvolume coordinates $(t, x_1, \ldots, x_{p-2})$ by V_{p-2}. The spatial volume of the Dp-brane with worldvolume coordinates $(t, x_1, \ldots, x_p)$ is then $V_p = V_{p-2}V_2$. The charge density of the Dp-brane in the bound state system is given by

$$Q_p = \frac{1}{2\kappa^2}\int_{\Omega_{8-p}} *F_{p+2} = \frac{(7-p)\Omega_{8-p}\cos\theta}{2\kappa^2 g} r_0^{7-p}\sinh\alpha\cosh\alpha\,, \tag{2.3}$$

where $2\kappa^2 = (2\pi)^7\alpha'^4$ is the gravity constant in ten dimensions and Ω_{8-p} is the volume of a unit $(8-p)$-sphere:

$$\Omega_{8-p} = \frac{2\pi^{(9-p)/2}}{\Gamma[(9-p)/2]} = \frac{4\pi \cdot \pi^{(7-p)/2}}{(7-p)\Gamma[(7-p)/2]} . \tag{2.4}$$

The D$(p-2)$-brane charge density on its worldvolume is

$$Q_{p-2} = \frac{1}{2\kappa^2}\int_{V_2\times\Omega_{8-p}} *F_p = \frac{(7-p)\Omega_{8-p}V_2\sin\theta}{2\kappa^2 g} r_0^{7-p}\sinh\alpha\cosh\alpha . \tag{2.5}$$

In fact the D$(p-2)$-brane charge density on the worldvolume of Dp-brane is

$$\tilde{Q}_{p-2} = \frac{Q_{p-2}}{V_2} . \tag{2.6}$$

According to the charge quantization rule $Q_p = T_p N_p$ in terms of the tension of the Dp-branes, we can obtain the number N_p of the Dp-branes and N_{p-2} of the D$(p-2)$-branes in the bound states. Defining $\tilde{R}^{7-p} = r_0^{7-p}\sinh\alpha\cosh\alpha$, we have the relation between the number N_p of Dp-branes and N_{p-2} of D$(p-2)$-branes:

$$\tilde{R}^{7-p} = N_p\frac{2\kappa^2 g T_p}{(7-p)\Omega_{8-p}\cos\theta} = N_{p-2}\frac{2\kappa^2 g T_{p-2}}{(7-p)\Omega_{8-p}V_2\sin\theta} , \tag{2.7}$$

where the tensions T_p and T_{p-2} have a unified expression as $T_p = (2\pi)^{-p}(\alpha')^{-(p+1)/2}$.

From (2.3) and (2.5), we can see that the asymptotic value $\tan\theta$ of the B-field has the following relation to the charges of the Dp- and D$(p-2)$-branes:

$$\tan\theta = \frac{\tilde{Q}_{p-2}}{Q_p} = \frac{1}{V_2}\frac{Q_{p-2}}{Q_p} = \frac{1}{V_2}\frac{T_{p-2}}{T_p}\frac{N_{p-2}}{N_p} . \tag{2.8}$$

The solution (2.1) has the event horizon at $r = r_0$ and hence has the associated thermodynamics. A standard calculation gives us the ADM mass M, Hawking temperature T and entropy S of the black configuration:

$$\begin{aligned} M &= \frac{(8-p)\Omega_{8-p}V_p r_0^{7-p}}{2\kappa^2 g^2}\left(1+\frac{7-p}{8-p}\sinh^2\alpha\right) \\ T &= \frac{7-p}{4\pi r_0\cosh\alpha} \\ S &= \frac{4\pi\Omega_{8-p}V_p}{2\kappa^2 g^2} r_0^{8-p}\cosh\alpha . \end{aligned} \tag{2.9}$$

It is somewhat surprising that these thermodynamic quantities are completely the same as those for the Dp-branes without B-field, just as noticed in [2, 5, 6, 7, 9]. The conclusion remains valid even if the angular rotation is introduced [9]. Here we focus on another aspect of the thermodynamics of this black configuration: these thermodynamic quantities are all independent of the parameter θ. As pointed out above, the parameter θ characterizes the interpolation of the solution between the black Dp-brane without B-field and the black D$(p-2)$-brane with two smeared coordinates and zero B-field (a nonvanishing constant B-field along the directions transverse to the D$(p-2)$-branes can be gauged away). Thus these thermodynamic quantities are also those of the black D$(p-2)$-branes with two smeared coordinates and no B-field. Therefore there is the thermodynamic equivalence not only between black Dp-branes with B-field and those without B-field, but also between the black Dp-branes with B-field and black D$(p-2)$-branes with two smeared coordinates and no B-field. This fact is important in the following discussions.

The thermodynamic quantities (2.9) with the charges (2.3) and (2.5) satisfy the first law of black hole thermodynamics as expected:

$$\begin{aligned} dM &= TdS + \mu_p dq_p + \mu_{p-2} dq_{p-2} \\ &= TdS + \mu_p V_p T_p dN_p + \mu_{p-2} V_{p-2} T_{p-2} dN_{p-2}\,, \end{aligned} \tag{2.10}$$

where μ_p and μ_{p-2} are the chemical potentials corresponding to the total charges $q_p = Q_p V_p$ and $q_{p-2} = V_{p-2} Q_{p-2}$, respectively,

$$\mu_p = \frac{\cos\theta \tanh\alpha}{g}\,, \qquad \mu_{p-2} = \frac{\sin\theta \tanh\alpha}{g}\,. \tag{2.11}$$

In addition, we notice that in the extremal limit (by taking $r_0 \to 0$ and $\alpha \to \infty$, but keeping $\tilde{R}^{7-p}$ constant),

$$M^2_{\rm ext.} = q_p^2 + q_{p-2}^2\,, \tag{2.12}$$

which indicates that the bound state (D$(p-2)$, Dp) is a nonthreshold one.

Now we turn to the field theory limit (decoupling limit) of the bound state solution (2.1), in which the gravity decouples from the field theory on the worldvolume of Dp-branes. Following [2, 5, 11], in the decoupling limit

$$\begin{aligned} \alpha' \to 0: \quad & \tan\theta = \frac{\tilde{b}}{\alpha'}\,, \quad r = \alpha' u\,, \quad r_0 = \alpha' u_0\,, \\ & g = \tilde{g}\alpha'^{(5-p)/2}\,, \quad x_{0,1,\ldots,p-2} = \tilde{x}_{0,1,\ldots,p-2}\,, \quad x_{p-1,p} = \frac{\alpha'}{\tilde{b}}\tilde{x}_{p-1,p}\,, \end{aligned} \tag{2.13}$$

and $\tilde{b}$, $\tilde{g}$, u, u_0, and $\tilde{x}_\mu$ held fixed, one has the following decoupling limit solution:

$$\begin{aligned} ds^2 &= \alpha' \left[\left(\frac{u}{R}\right)^{(7-p)/2} \left(-\tilde{f}dt^2 + d\tilde{x}_1^2 + \cdots + d\tilde{x}_{p-2}^2 + \tilde{h}(d\tilde{x}_{p-1}^2 + d\tilde{x}_p^2) \right) + \right. \\ &\quad \left. + \left(\frac{R}{u}\right)^{(7-p)/2} \left(\tilde{f}^{-1}du^2 + u^2 d\Omega_{8-p}^2 \right) \right], \\ e^{2\phi} &= \tilde{g}^2 \tilde{b}^2 \tilde{h} \left(\frac{R}{u}\right)^{(7-p)(3-p)/2}, \qquad B_{p-1,p} = \frac{\alpha'}{\tilde{b}} \frac{(au)^{7-p}}{1+(au)^{7-p}}\,, \end{aligned} \tag{2.14}$$

where the corresponding RR fields are not exposed explicitly,

$$\tilde{f} = 1 - \left(\frac{u_0}{u}\right)^{7-p}, \qquad \tilde{h} = \frac{1}{1+(au)^{7-p}}, \qquad a^{7-p} = \frac{\tilde{b}^2}{R^{7-p}}, \tag{2.15}$$

and

$$R^{7-p} = \frac{1}{2}(2\pi)^{6-p}\pi^{-(7-p)/2}\Gamma[(7-p)/2]\tilde{g}\tilde{b}N_p\,. \tag{2.16}$$

According to the generalized AdS/CFT correspondence, the solution (2.14) is the dual gravity description of the noncommutative gauge field theory with gauge group $\mathrm{U}(N_p)$ in $(p+1)$ dimensions [1, 2, 5, 6, 9]. When $a = 0$, the solution (2.14) reduces to the usual decoupling limit solution of Dp-branes without B-field [24]. This implies that the noncommutativity effect is weak in field theories for $au \ll 1$ or at long distance.

In the decoupling limit, the thermal excitations above the extremality have the energy E, temperature T and entropy S:

$$\begin{aligned} E &= \frac{(9-p)\Omega_{8-p}\tilde{V}_p}{2(2\pi)^7(\tilde{g}\tilde{b})^2}u_0^{7-p} \\ T &= \frac{7-p}{4\pi}R^{-\frac{7-p}{2}}u_0^{\frac{5-p}{2}} \\ S &= \frac{2\Omega_{8-p}\tilde{V}_p}{(2\pi)^6(\tilde{g}\tilde{b})^2}R^{\frac{7-p}{2}}u_0^{(9-p)/2}\,. \end{aligned} \tag{2.17}$$

Here $\tilde{V}_p = V_{p-2}\tilde{V}_2$ is the spatial volume of the Dp-brane after taking the decoupling limit (2.14), and $\tilde{V}_2 = V_2\tilde{b}^2/\alpha'^2$ is the area of the torus. Using (2.17), one finds the free energy, defined as $F = E - TS$, of the thermal excitations:

$$\begin{aligned} F &= -\frac{(5-p)\Omega_{8-p}\tilde{V}_p}{2(2\pi)^7(\tilde{g}\tilde{b})^2}u_0^{7-p} \\ &= -\frac{\Omega_{8-p}V_{p-2}\tilde{V}_2}{(2\pi)^7\tilde{g}^2\tilde{b}^2}\frac{5-p}{2}\left(\frac{4\pi}{7-p}\right)^{\frac{2(7-p)}{5-p}}R^{\frac{(7-p)^2}{5-p}}T^{\frac{2(7-p)}{5-p}}, \end{aligned} \tag{2.18}$$

in terms of the temperature. In the decoupling limit, the numbers of two kinds of branes are constants. Hence the first law of thermodynamics becomes

$$dE = TdS \qquad \text{and} \qquad dF = -SdT\,. \tag{2.19}$$

In the spirit of the generalized AdS/CFT correspondence, the thermodynamics of the thermal excitations should be equivalent to that of the corresponding non-commutative gauge fields at finite temperature T in the large-N and strong 't Hooft coupling limit. We notice that these thermodynamic quantities, after rescaling the string coupling constant as $\tilde{g}\tilde{b} = \hat{g}$, are exactly the same as those of the black Dp-branes without B-field in the decoupling limit. This means that in this supergravity approximation, the thermodynamics of the large-N noncommutative and ordinary gauge field theories both in $(p+1)$ dimensions are equivalent to each other. This also implies that in the planar limit, the number of the degrees of freedom in the noncommutative gauge theories coincides with that in the ordinary field theories not only in the weak coupling limit [25], but also in the strong coupling limit [2, 5, 6, 9].

Now let us recall the fact that the thermodynamics (2.9) of the black Dp-branes with nonzero B-field is the same as that of the black D$(p-2)$-branes with two smeared coordinates and zero B field. In the decoupling limit (2.14), the solution (2.14) is described by the quantities of Dp-branes. For instance, R^{7-p} is proportional to the number N_p of the coinciding Dp-branes (see (2.16)). In fact the "radius" R can also be expressed by quantities of D$(p-2)$-branes. Using (2.8), we obtain

$$R^{7-p} = \frac{1}{2}(2\pi)^{6-p}\pi^{-(7-p)/2}\Gamma[(7-p)/2]\tilde{g}\tilde{b}N_{p-2} \times \frac{(2\pi)^2\tilde{b}}{\tilde{V}_2}\,. \tag{2.20}$$

In the decoupling limit, we are thus led to the relation between the numbers of Dp- and D$(p-2)$-branes:

$$\tan\theta = \frac{\tilde{b}}{\alpha'} = \frac{(2\pi)^2\tilde{b}^2}{\alpha'\tilde{V}_2}\frac{N_{p-2}}{N_p} \qquad \Longrightarrow \qquad \frac{N_{p-2}}{N_p} = \frac{\tilde{V}_2}{(2\pi)^2\tilde{b}}. \tag{2.21}$$

Because $\tilde{V}_2$ and $\tilde{b}$ can be kept finite, we can thus conclude that in the decoupling limit of the Dp-branes with NS B-field, the number of the D$(p-2)$-branes can be kept finite. This looks different from the claim by Lu and Roy [11] where they conclude that the number of the D$(p-2)$-branes becomes infinity in the decoupling limit. This is so because they take a little different decoupling limit and there $\tilde{x}_{p-1}$ and $\tilde{x}_p$ are infinitely extended. Mathematically our decoupling limit becomes the same as theirs by taking $\tilde{V}_2 \to \infty$ but keeping $N_{p-2}/\tilde{V}_2 = N_p/(2\pi)^2\tilde{b}$ finite.

In the decoupling limit (2.14), $\tan\theta \to \infty$ as $\alpha' \to 0$. If we set $\theta = \pi/2$, the solution (2.1) reduces to the black D$(p-2)$-brane with two smeared coordinates and zero B-field after gauging away the constant value. This shift of the constant B is allowed in the large-N limit [25].[1] The decoupling limit solution (2.14) for the black Dp-brane with NS B-field is thus expected to be related with the solution of black D$(p-2)$-brane with two smeared coordinates and no B-field in the same decoupling limit. For our convenience, we rewrite the black D$(p-2)$-brane with two smeared coordinates:

$$\begin{aligned} ds^2 &= H^{-1/2}[-f dt^2 + dx_1^2 + \cdots + dx_{p-2}^2 + H(dx_{p-1}^2 + dx_p^2)] \\ &\quad + H^{1/2}(f^{-1}dr^2 + r^2 d\Omega_{8-p})\,, \\ e^{2\phi} &= g^2 H^{(5-p)/2}\,, \qquad A^{p-2}_{01\cdots(p-2)} = g^{-1}(H^{-1}-1)\coth\alpha\,, \qquad B_{p-1,p} = 0\,, \end{aligned} \tag{2.22}$$

[1] Alternatively, if one simply takes the usual D$(p-2)$-brane solution, there is no B-field.

where H and f are the same as those in (2.2). Note that here x_{p-1} and x_p are two smeared transverse coordinates for the D$(p-2)$-branes. In the decoupling limit (2.14), we reach

$$\begin{aligned} ds^2 &= \alpha' \left[\left(\frac{u}{R}\right)^{(7-p)/2} \left(-\tilde{f} dt^2 + d\tilde{x}_1^2 + \cdots + d\tilde{x}_{p-2}^2 + \frac{1}{(au)^{7-p}} (d\tilde{x}_{p-1}^2 + d\tilde{x}_p^2) \right) + \right. \\ &\quad \left. + \left(\frac{R}{u}\right)^{(7-p)/2} \left(\tilde{f}^{-1} du^2 + u^2 d\Omega_{8-p}^2 \right) \right], \\ e^{2\phi} &= \tilde{g}^2 \tilde{b}^{5-p} (au)^{(7-p)(p-5)/2}, \qquad B_{p-1,p} = 0, \end{aligned} \tag{2.23}$$

where $\tilde{f}$ and R^{7-p} are given in (2.15) and (2.20), respectively. When $\tilde{f} = 1$, the solution reduces to that given in [11]. Obviously for $au \gg 1$, the decoupling solution (2.14) of the Dp-brane with NS B-field is indeed equivalent to the decoupling limit solution (2.23) of the black D$(p-2)$-branes with two smeared coordinates and no NS B field, as noticed in [11]. (We will also discuss the equivalence from the thermodynamics point of view shortly.) Note that the coordinate u corresponds to an energy scale of worldvolume gauge field theories, and in (2.14) au reflects the noncommutative effect of gauge fields. It has been found that in order for the dual gravity description (2.14) of noncommutative gauge fields to be valid, $au \gg 1$ should be satisfied [11, 5], in which case N_p can be small and the noncommutativity effect is strong in the corresponding field theories.

We know that the solution (2.14) is a dual gravity description of a noncommutative super Yang-Mills theory with gauge group U(N_p) in $(p+1)$ dimensions. What is the field theory corresponding to the supergravity solution (2.23) for large au? To see this, let us note that the supergravity description (2.23) breaks down for large au since the effective size of the torus shrinks. Nevertheless, we can make a T-duality along the directions $\tilde{x}_{p-1}$ and $\tilde{x}_p$. We then obtain a usual decoupling limit solution of N_{p-2} coincident Dp-branes without B field [24]:

$$\begin{aligned} ds^2 &= \alpha' \left[\left(\frac{u}{R}\right)^{(7-p)/2} \left(-\tilde{f} dt^2 + d\tilde{x}_1^2 + \cdots + d\tilde{x}_{p-2}^2 + dx_{p-1}^2 + dx_p^2 \right) + \right. \\ &\quad \left. + \left(\frac{R}{u}\right)^{(7-p)/2} \left(\tilde{f}^{-1} du^2 + u^2 d\Omega_{8-p}^2 \right) \right] \\ e^{2\phi} &= \frac{(2\pi)^4 \tilde{g}^2 \tilde{b}^4}{\tilde{V}_2^2} \left(\frac{u}{R}\right)^{(7-p)(p-3)/2}, \qquad \tilde{B}_{p-1,p} = 0. \end{aligned} \tag{2.24}$$

This solution describes a $(p+1)$-dimensional ordinary super Yang-Mills theory with gauge group U(N_{p-2}) on the dual torus with area $\hat{V}_2 = (2\pi)^4 \tilde{b}^2 / \tilde{V}_2$. Since the dual torus is characterized by the periodicity $x_{p-1,p} \sim x_{p-1,p} + \sqrt{\hat{V}_2}$, the radii of the dual torus go to zero for $\tilde{V}_2 \to \infty$ and the $(p+1)$-dimensional ordinary super Yang-Mills theory then reduces to a $(p-1)$-dimensional one. This means that if the torus in (2.23) is very large ($\tilde{V}_2 \to \infty$), the solution is a dual gravity description of a $(p-1)$-dimensional ordinary super Yang-Mills theory with gauge group $\mathrm{U}(\infty)$. The $(p-1)$-dimensional theory has the Yang-Mills coupling constant

$$g_{\mathrm{YM}}^2 = (2\pi)^{p-4}\tilde{g}\,, \tag{2.25}$$

while the coupling constant of the $(p+1)$-dimensional noncommutative gauge field is $g_{\mathrm{YM}}^2 = (2\pi)^{p-2}\tilde{g}\tilde{b}$. Thus we reach the equivalence argued by Lu and Roy [11] between the noncommutative super Yang-Mills theory in $(p+1)$ dimensions and the ordinary one with gauge group $\mathrm{U}(\infty)$ in $(p-1)$ dimensions.

This equivalence can also be understood from a T-duality of the decoupling limit solution (2.14) for the Dp-branes with B-field. A usual T-duality transformation [39] is, however, not enough for this purpose since the resulting radius for the torus is not large after the usual T-duality in the presence of B field. This can be remedied if we use more general T-duality transformation $\mathrm{SL}(2,\mathbb{Z})$ [40] as described in refs. [15, 8]. The duality transformation

$$\rho \to \frac{a\rho + b}{c\rho + d}\,, \qquad \rho \equiv \frac{\tilde{V}_2}{(2\pi)^2\alpha'}\left(B_{p-1,p} + i\sqrt{G_{(p-1)(p-1)}G_{pp}}\right), \tag{2.26}$$

gives a dual solution

$$\begin{aligned} ds^2 &= \alpha'\left[\left(\frac{u}{R}\right)^{(7-p)/2}\left(-\tilde{f}dt^2 + d\tilde{x}_1^2 + \cdots + d\tilde{x}_{p-2}^2 + dx_{p-1}^2 + dx_p^2\right) + \right. \\ &\qquad \left. + \left(\frac{R}{u}\right)^{(7-p)/2}\left(\tilde{f}^{-1}du^2 + u^2 d\Omega_{8-p}^2\right)\right], \\ e^{2\phi} &= \frac{(2\pi)^4\tilde{g}^2\tilde{b}^4}{\tilde{V}_2^2}\left(\frac{u}{R}\right)^{(7-p)(p-3)/2}, \qquad \tilde{B}_{p-1,p} = \frac{\alpha'}{\tilde{b}}\,, \end{aligned} \tag{2.27}$$

by choosing $c = -1$ and $d = \tilde{V}_2/(2\pi)^2\tilde{b}$ when the latter is an integer. Note that $d = N_{p-2}/N_p$ must be a rational number. If this is not an integer, after some steps of Morita equivalence transformation following [8], one can reach a solution like (2.27). For $p = 3$ and $\tilde{f} = 1$, the solution (2.27) reduces to the case discussed in [15, 8]. Note that the solution (2.27) is the same as (2.24) except that the former has a nonvanishing constant B-field while the latter has zero B-field. The solution (2.27) describes a twisted ordinary super Yang-Mills theory with gauge group $\mathrm{U}(N_{p-2})$ in $(p+1)$ dimensions, living on the dual torus with area $\hat{V}_2 = (2\pi)^4\tilde{b}^2/\tilde{V}_2$. For $\tilde{V}_2 \to \infty$, however, the theory reduces to a $(p-1)$-dimensional ordinary super Yang-Mills theory. Therefore we again arrive at the conclusion that the $(p+1)$-dimensional noncommutative gauge field is equivalent to an ordinary gauge field with gauge group $\mathrm{U}(\infty)$ in $(p-1)$ dimensions for $\tilde{V}_2 \to \infty$, from the point of view of dual gravity description.

Next let us address another evidence to render support of the above equivalence from the viewpoint of thermodynamics. We find that the thermodynamics of decoupling limit solution (2.23) of the D$(p-2)$-branes is completely the same as those in (2.17). The worldvolume theory of the (D$(p-2)$, Dp) bound states (2.1) (or N_p coinciding Dp-branes with B-field) is a noncommutative gauge field theory with gauge group U(N_p) in $(p+1)$ dimensions with two dimensions compactified on a torus, while the worldvolume theory is an ordinary gauge field theory with the same gauge group U(N_p) if the NS B-field is absent. The above equivalence of the descriptions of the (D$(p-2)$, Dp) bound states implies that the bound states can also be described by ordinary gauge field theories in $(p+1)$ dimensions. Moreover, the equivalence is valid also between the $(p+1)$-dimensional noncommutative U(N_p) theory and the $(p-1)$-dimensional U(∞) ordinary theory in the limit $\tilde{V}_2 \to \infty$ for the reason described above. For finite volume, the equivalence is between the $(p+1)$-dimensional noncommutative U(N_p) gauge field and a (twisted) ordinary U(N_{p-2}) gauge field with the relation (2.21), the latter living on a dual torus. In this case, the Yang-Mills coupling constant is

$$g^2_{\rm YM} = \frac{(2\pi)^p \tilde{g}\tilde{b}^2}{\tilde{V}_2}\,, \tag{2.28}$$

for the $(p+1)$-dimensional ordinary gauge field theory. As a self-consistency check, one may find that the coupling constant (2.25) can also be obtained from (2.28) after a trivial dimensional reduction. In the following sections we will further discuss the equivalence of the descriptions and the relation (2.21) from the point of view of probe branes.

3. The static probes: thermodynamics

The D-brane probe is a useful tool to explore the structure of D-brane bound states (see for example [41]–[47] and references therein). Recently the D-brane probes have been used to check a certain aspect of AdS/CFT correspondence [48]–[52]. In this section we consider the static interaction potentials (thermodynamics) of two kinds of probes in the background of (D$(p-2)$, Dp) bound states. One of them is a bound state probe consisting of D$(p-2)$- and Dp-branes, or Dp-brane probe with B-fields, or noncommutative Dp-brane probe; the other is a D$(p-2)$-brane probe. Let us first discuss the noncommutative Dp-brane probe.

3.1 A noncommutative Dp-brane probe

Due to the presence of the nonvanishing NS B-field in the noncommutative Dp-brane probe, the probe should have the following action:

$$S_p = -T_p \int d^{p+1}x e^{-\phi}\sqrt{-\det(G_{ab}+\mathcal{F}_{ab})} + T_p \int A^p + T_p \int A^{p-2}\wedge \mathcal{F}\,, \tag{3.1}$$

where $\mathcal{F}_{ab} = (2\pi\alpha')F_{ab} + B_{ab}$, F_{ab} is the gauge field strength on the worldvolume of the Dp-brane and B_{ab} is the NS B-field. Here we set $F_{ab} = 0$. As demonstrated in (2.1), the occurrence of the NS B-field in the Dp-branes is always accompanied by the appearance of D$(p-2)$-branes in the system, forming (D$(p-2)$, Dp) bound states. The probe can be regarded as a bound state probe consisting of D$(p-2)$- and Dp-branes for the following reasons: (1) The probe has the tension $T_p\sqrt{1+\tan^2\theta}$, the same as the (D$(p-2)$, Dp) bound states; (2) We will see shortly that the static interaction potential of the probe vanishes in the background of the nonthreshold (D$(p-2)$, Dp) bound states; (3) Except for the source A^p of the Dp-branes, the source A^{p-2} of the D$(p-2)$-branes also occurs in the action (3.1); (4) From its thermodynamics we will obtain further evidence of this interpretation. Because of the presence of the NS B-field, we can also view the probe as a noncommutative Dp-brane probe.

Substituting the solution (2.1) into the probe action (3.1), one has

$$S_p = -\frac{T_pV_p}{g\cos\theta}\int d\tau H^{-1}\left[\sqrt{f} - 1 + H_0 - H\right], \tag{3.2}$$

where we have subtracted a constant potential at spatial infinity and

$$H_0 = 1 + \left(\frac{\tilde{R}}{r}\right)^{7-p} = 1 + \frac{r_0^{7-p}\sinh\alpha\cosh\alpha}{r^{7-p}}. \tag{3.3}$$

In the extremal limit ($f=1$), the static interaction potential vanishes, which verifies that the probe is a bound state of D$(p-2)$-branes and Dp-branes because the source is nonthreshold bound states of D$(p-2)$- and Dp-branes. Unless the probe is such a kind of bound state, the static potential will no longer vanish. In the non-extremal background, the interaction potential always exists.

Now suppose the probe is moved from spatial infinity to the horizon of the source [49]. From (3.2) we can obtain the potential difference (which is just the potential at the horizon because we have set the potential zero at spatial infinity)

$$U_p|_{r=r_0} = \frac{T_pV_p}{g\cos\theta}\left(1-\tanh\alpha\right). \tag{3.4}$$

Since the tension of the probe is $T_p\sqrt{1+\tan^2\theta}$, it is easy to show that the first term is just the mass of the probe because

$$m_p = \frac{T_pV_p}{g}\sqrt{1+\tan^2\theta} = \frac{T_pV_p}{g\cos\theta}. \tag{3.5}$$

The second term in (3.4) has the following interpretation. Let us denote the numbers of Dp-branes and D$(p-2)$-branes in the probe by δN_p and δN_{p-2}, respectively. We then have

$$\begin{aligned}\mu_p V_p T_p \delta N_p + \mu_{p-2} V_{p-2} T_{p-2} \delta N_{p-2} &= \frac{T_p V_p}{g \cos\theta}\left[\cos^2\theta + \sin\theta\cos\theta \frac{V_{p-2}T_{p-2}}{V_p T_p}\frac{\delta N_{p-2}}{\delta N_p}\right]\delta N_p \\ &= \frac{T_p V_p}{g\cos\theta}\tanh\alpha \delta N_p\,, \end{aligned} \tag{3.6}$$

where in obtaining the second line we have used the fact that the form of the tension of the probe implies that the number of the branes obey $\delta N_{p-2}/\delta N_p = \tan\theta V_p T_p/(V_{p-2}T_{p-2})$. When $\delta N_p = 1$, this quantity (3.6) gives the second term in (3.4). This process satisfies the first law of thermodynamics (2.10). In fact eq. (3.4) reduces to (2.10) with $dM = m_p$, $U_p|_{r=r_0} = TdS$ and (3.6). It follows that the potential of the probe is converted into heat energy and is absorbed by the source when the probe moves to the horizon from spatial infinity. Our calculation also shows that the probe is a bound state of D$(p-2)$- and Dp-branes.

Now we consider the decoupling limit of the static probe action. In this limit, we obtain

$$S_p = -\frac{V_{p-2}\tilde{V}_2}{(2\pi)^p \tilde{g}\tilde{b}}\int d\tau \left(\frac{u}{R}\right)^{7-p}\left[\sqrt{\tilde{f}} - 1 + \frac{u_0^{7-p}}{2u^{7-p}}\right]. \tag{3.7}$$

From the action we can also obtain the free energy of the probe at the temperature T, which is just the Hawking temperature of the source given in (2.17):

$$F_p = \frac{V_{p-2}\tilde{V}_2}{(2\pi)^p \tilde{g}\tilde{b}}\left(\frac{u}{R}\right)^{7-p}\left[\sqrt{\tilde{f}} - 1 + \frac{u_0^{7-p}}{2u^{7-p}}\right]. \tag{3.8}$$

In the generalized AdS/CFT correspondence, the thermodynamics (2.17) is equivalent to that of noncommutative supersymmetric gauge fields with gauge group U(N_p) in the large-N and strong coupling limit (within the valid regime of dual gravity description). From the viewpoint of field theory, the thermodynamics corresponds to that of the gauge field in the Higgs branch, where the gauge group is not broken and hence the vacuum expectation values of scalars vanish. According to the interpretation of a D-brane probe action [42], the thermodynamics of a D-brane probe can be regarded as the thermodynamics of the supersymmetric gauge field in the Coulomb branch (Higgs phase) [48, 50], in which the original gauge group is broken; some vacuum expectation values of scalar fields do not vanish; and the distance u between the probe and the source can be viewed as a energy scale in the gauge fields. This interpretation of thermodynamics of D-brane probe turns out to be consistent with the expectation on the field theory side [48, 50].

Since the rescaled string coupling is $\hat{g} = \tilde{g}\tilde{b}$, we find from (3.8) that in the decoupling limit, the free energy of a noncommutative Dp-brane probe in the noncommutative Dp-brane background ((D$(p-2)$, Dp) bound states) is exactly the same as that of an ordinary Dp-brane probe in the Dp-brane background without NS B-field (for the interaction potential of the latter see [50]). Thus, in the supergravity approximation, the thermodynamics of noncommutative gauge fields remains the same as the ordinary case both in the Higgs and Coulomb branches. We thus conclude that in the large-N limit, the number of the degrees of freedom of noncommutative gauge fields coincides with the ordinary case, not only in the weak coupling limit, but also in the strong coupling limit.

Now we consider the difference between the interaction potentials (free energy) of the probe at the infinity and at the horizon in the decoupling limit, and compare it with the asymptotically flat case already discussed before. Note that the free energy still vanishes at the infinity ($u \to \infty$) as can be seen in (3.8). Thus the difference in the free energies is just the free energy of the probe at the horizon u_0:

$$\begin{aligned} F_p|_{u=u_0} &= -\frac{V_{p-2}\tilde{V}_2}{2(2\pi)^p\tilde{g}\tilde{b}}\left(\frac{u_0}{R}\right)^{7-p} \\ &= -\frac{V_{p-2}\tilde{V}_2}{2(2\pi)^p\tilde{g}\tilde{b}}\left(\frac{4\pi RT}{7-p}\right)^{\frac{2(7-p)}{5-p}}. \end{aligned} \tag{3.9}$$

We find that the free energy of the probe at the horizon (3.9) has the relation with that of the source (2.18) as

$$F_p|_{u=u_0} = \frac{dF}{dN_p}\delta N_p\,, \tag{3.10}$$

with $\delta N_p = 1$. Note that we are considering a Dp-brane with B-field in the background of N_p Dp-branes. Therefore in the large N_p limit (that is $N_p \gg 1$), the probe free energy is expected to be

$$F_p|_{u=u_0} \approx F(N_p+1) - F(N_p)\,, \tag{3.11}$$

consistent with (3.10). This relation supports the argument that the non-extremal Dp-branes is *located* at the horizon. (In the next section we will further show that indeed Dp-branes can be located at the horizon). This also supports the interpretation of thermodynamics of probe branes given in [48, 50], because the probe brane at the horizon of the source can be considered to coincide with source branes and the gauge symmetry is restored and the probe brane can be seen as a part of the source branes in this case.

3.2 A D$(p-2)$-brane probe

Let us next consider a D$(p-2)$-brane probe. The action of a D$(p-2)$-brane in the background of the (D$(p-2)$, Dp) bound states (2.1) is

$$S_{p-2} = -T_{p-2}\int d^{p-1}x e^{-\phi}\sqrt{-\det G_{ab}} + T_{p-2}\int A^{p-2}\,. \tag{3.12}$$

Substituting the background solution (2.1) into the action yields, for a static probe,

$$S_{p-2} = -\frac{T_{p-2}V_{p-2}}{g}\int d\tau H^{-1}\left[H^{1/2}h^{-1/2}\sqrt{f} - (1-H_0)\sin\theta - H\right], \tag{3.13}$$

where we have also subtracted a constant potential so that the interaction potential vanishes at spatial infinity. Note that the static interaction potential is quite different from that of the noncommutative Dp-brane probe in the same background (2.1). Indeed the potential (3.13) does not vanish even when the background is sent to the extremal limit of the solution. This is consistent with the fact that the source is a nonthreshold bound state consisting of D$(p-2)$- and Dp-branes.

As in the noncommutative Dp-brane probe, let us first consider the asymptotically flat background. In this case, we find that the deference between the potentials at the spatial infinity and at the horizon is

$$U_{p-2}|_{r=r_0} = \frac{V_{p-2}T_{p-2}}{g}(1-\sin\theta\tanh\alpha). \tag{3.14}$$

The first term is the mass of the probe $m_{p-2} = T_{p-2}V_{p-2}/g$, while the second term is equal to $\mu_{p-2}V_{p-2}T_{p-2}\delta N_{p-2}$ with $\delta N_{p-2} = 1$ since we are considering a probe D$(p-2)$-brane. Therefore the D$(p-2)$-brane probe falling to the horizon from the spatial infinity satisfies the first law (2.10) again with $dM = m_{p-2}$, $\delta N_{p-2} = 1$, and $\delta N_p = 0$. The potential difference of the D$(p-2)$-brane probe is converted into heat energy at the horizon and thereby is absorbed by the source.

Comparing (3.2) and (3.13), *a priori* one may think that they are quite different and there seems to be no relation between them. Actually once the decoupling limit (2.14) is taken, one may find that there is a close relation between (3.2) and (3.13). In the decoupling limit, the action of the D$(p-2)$-brane probe becomes

$$S_{p-2} = -\frac{V_{p-2}}{(2\pi)^{p-2}\tilde{g}}\int d\tau\left(\frac{u}{R}\right)^{7-p}\left[\sqrt{\frac{1+(au)^{7-p}}{(au)^{7-p}}}\sqrt{\tilde{f}} - 1 + \frac{u_0^{7-p}}{2u^{7-p}}\right]. \tag{3.15}$$

As mentioned above, the validity of the dual gravity description of gauge field theories requires $au \gg 1$. The above action then reduces to

$$S_{p-2} = -\frac{V_{p-2}}{(2\pi)^{p-2}\tilde{g}}\int d\tau\left(\frac{u}{R}\right)^{7-p}\left[\sqrt{\tilde{f}} - 1 + \frac{u_0^{7-p}}{2u^{7-p}}\right]. \tag{3.16}$$

The corresponding free energy of the probe at the distance u is

$$F_{p-2} = -\frac{V_{p-2}}{(2\pi)^{p-2}\tilde{g}}\left(\frac{u}{R}\right)^{7-p}\left[\sqrt{\tilde{f}} - 1 + \frac{u_0^{7-p}}{2u^{7-p}}\right]. \tag{3.17}$$

At the horizon u_0 it is

$$\begin{aligned} F_{p-2}|_{u=u_0} &= -\frac{V_{p-2}}{2(2\pi)^{p-2}\tilde{g}}\left(\frac{u_0}{R}\right)^{7-p} \\ &= -\frac{V_{p-2}}{2(2\pi)^{p-2}\tilde{g}}\left(\frac{4\pi RT}{7-p}\right)^{\frac{2(7-p)}{5-p}}. \end{aligned} \tag{3.18}$$

Comparing the free energy (3.18) with the one (3.9) of the noncommutative Dp-brane probe, one may find that they are the same up to a different prefactor. Consequently the free energy of δN_{p-2} D$(p-2)$-branes is the same as that of δN_p noncommutative Dp-branes if the relation

$$\frac{\delta N_{p-2}}{\delta N_p} = \frac{\tilde{V}_2}{(2\pi)^2 \tilde{b}}\,, \tag{3.19}$$

is obeyed. We see that this relation coincides with eq. (2.21). Note that the relation (2.21) is derived from the two equivalent descriptions of the bound state source, while (3.19) is obtained from the equivalence of probes in the same background. In other words, the probe consisting of δN_p noncommutative Dp-branes is equivalent to the probe consisting of δN_{p-2} D$(p-2)$-branes since they get the same response in the same background. Furthermore, we find

$$F_{p-2}|_{u=u_0} = \frac{dF}{dN_{p-2}}\delta N_{p-2}\,, \tag{3.20}$$

with $\delta N_{p-2} = 1$. When $N_{p-2} \gg 1$, once again, we have

$$F_{p-2}|_{u=u_0} \approx F(N_{p-2}+1) - F(N_{p-2})\,. \tag{3.21}$$

This implies that from the point of view of the D$(p-2)$-brane probe, the bound state source (D$(p-2)$, Dp) can be viewed as N_{p-2} coincident D$(p-2)$-branes with two smeared coordinates and zero NS B-field, while from the noncommutative Dp-brane probe, the source is N_p coincident Dp-branes with nonzero NS B-field. Therefore from the thermodynamics of probe branes, we again find that the bound states (D$(p-2)$, Dp) have two equivalent descriptions.

4. The dynamical probes: absorbing or scattering

In this section we will consider the dynamical aspect of the two kinds of probes discussed in the previous section.

4.1 The noncommutative Dp-brane probe

To investigate the dynamics of the probe, it is convenient to take static gauge: $\tau = t$, x_i act just as the worldvolume coordinates and other transverse coordinates depend only on τ. In the background (2.1), the action (3.1) of the noncommutative Dp-brane probe reduces to

$$S_p = -\frac{T_p V_p}{g\cos\theta}\int d\tau H^{-1}\left[\sqrt{f - H(f^{-1}\dot{r}^2 + r^2\dot{\Omega}^2_{8-p})} - 1 + H_0 - H\right], \tag{4.1}$$

where an overdot denotes the derivative with respect to τ. In the decoupling limit, the action becomes

$$S_p = -m_p\int d\tau \left(\frac{u}{R}\right)^{7-p}\left[\sqrt{\tilde{f} - \left(\frac{R}{u}\right)^{7-p}\left(\tilde{f}^{-1}\dot{u}^2 + u^2\dot{\Omega}^2_{8-p}\right)} - 1 + \frac{u_0^{7-p}}{2u^{7-p}}\right], \tag{4.2}$$

where $m_p = V_{p-2}\tilde{V}_2/[(2\pi)^p\tilde{g}\tilde{b}]$ is the mass of the probe. With the definition of the isotropic coordinates

$$\tilde{f}^{-1}du^2 + u^2 d\Omega_{8-p}^2 = u^2\rho^{-2}(d\rho^2 + \rho^2 d\Omega_{8-p}^2)\,, \tag{4.3}$$

where

$$u^{7-p} = \rho^{7-p}\left(1 + \frac{u_0^{7-p}}{4\rho^{7-p}}\right)^2, \tag{4.4}$$

we can define the velocity of the probe as

$$\tilde{f}^{-1}\dot{u}^2 + u^2\dot{\Omega}_{8-p}^2 \equiv u^2\rho^{-2}v^2\,. \tag{4.5}$$

In the low velocity and long distance approximation, expanding (4.2) yields

$$S_p = \int d\tau \left[\frac{1}{2}m_p v^2 - \mathcal{V}(\rho, v) + \mathcal{O}\left(\frac{1}{\rho^{2(7-p)}}\right)\right], \tag{4.6}$$

where the interaction potential $\mathcal{V}$ is

$$\mathcal{V}(\rho, v) = -m_p\frac{u_0^{7-p}}{\rho^{7-p}}\left\{\frac{9-p}{4(7-p)}v^2 + \frac{1}{8}\left[\left(\frac{u_0}{R}\right)^{7-p} + \left(\frac{R}{u_0}\right)^{7-p} v^4\right]\right\}. \tag{4.7}$$

When $\theta = 0$, the background (2.1) reduces to the black Dp-brane without B and the probe action (4.1) to the one for a Dp-brane without B-field. The interaction potential (4.7) therefore is also the one for a Dp-brane probe in other Dp-brane background. Up to order v^4, it can be seen that (4.7) reduces to the result in [42] for a Dp-brane in other Dp-brane background. This interaction potential can be reproduced in the one-loop calculation of the noncommutative field theory. Using the relation between the phase shift of scattering and the potential [44]

$$\delta(\rho, v) = -\int_0^\infty d\tau \mathcal{V}[\rho(\tau), v]\,, \qquad \rho^2(\tau) = \rho^2 + v^2\tau^2\,, \tag{4.8}$$

we obtain the phase shift of the probe. Writing $\mathcal{V}(\rho, v) = \lambda(v)\rho^{-(7-p)}$, we find

$$\delta(\rho, v) = -\frac{B\left(\frac{1}{2}, \frac{6-p}{2}\right)}{2v\rho^{6-p}}\lambda(v)\,, \tag{4.9}$$

where B is the beta function.

Next we discuss the classical motion of the noncommutative Dp-brane probe near the horizon of the source (that is, in the decoupling limit). For simplicity, let us consider the case of the probe with angular momentum only in a single direction (say, ϕ-direction). Following refs. [46, 47], from (4.2) we obtain its angular momentum

$$L = \frac{m_p u^2\dot{\phi}}{\sqrt{\tilde{f} - (R/u)^{7-p}\left(\tilde{f}^{-1}\dot{u}^2 + u^2\dot{\phi}^2\right)}}\,. \tag{4.10}$$

The energy of the probe is

$$E = \frac{m_p (u/R)^{7-p} \tilde{f}}{\sqrt{\tilde{f} - (R/u)^{7-p} \left(\tilde{f}^{-1} \dot{u}^2 + u^2 \dot{\phi}^2 \right)}} - m_p \left(\frac{u}{R} \right)^{7-p} \left(1 - \frac{u_0^{7-p}}{2u^{7-p}} \right) . \tag{4.11}$$

With the relation

$$E = \frac{1}{2} m_p \dot{u}^2 + V(u) , \tag{4.12}$$

one may obtain an effective central potential of the radial motion of the probe

$$V(u) = E \left[1 - \frac{m_p \tilde{f}^2}{2E} \left(\frac{u}{R} \right)^{7-p} \left(1 - \frac{\tilde{f}}{\mathcal{A}^2} \right) \right] + \frac{L^2 \tilde{f}^3}{2 m_p u^2 \mathcal{A}^2} , \tag{4.13}$$

where

$$\mathcal{A} = 1 - \frac{u_0^{7-p}}{2u^{7-p}} + \frac{E}{m_p} \left(\frac{R}{u} \right)^{7-p} . \tag{4.14}$$

The qualitative features of the motion of the probe can be understood by finding out the turning points, at which $\dot{u} = 0$.

Let us first discuss the case of extremal background (or in the background of the nonthreshold bound state (D$(p-2)$, Dp). One has $\tilde{f} = 1$. From the effective central potential one can see that if the angular momentum vanishes, there is no turning point for the probe. It follows that the probe will be captured by the source. When the angular momentum does not vanish, the potential (4.13) reduces to

$$V(u) = E \left\{ 1 - \frac{1}{2} \left(\frac{u}{u_*} \right)^{7-p} \left[1 - \frac{1}{\left(1 + (u_*/u)^{7-p} \right)^2} \right] \right\} + \frac{E u_{**}^2}{2u^2 \left(1 + (u_*/u)^{7-p} \right)^2} , \tag{4.15}$$

where we have introduced two characterizing lengths

$$u_* = R \left(\frac{E}{m_p} \right)^{1/(7-p)} , \qquad u_{**} = L \left(\frac{1}{m_p E} \right)^{1/2} . \tag{4.16}$$

The turning point satisfies the following equation:

$$2 + \left(\frac{u_*}{u_c} \right)^{7-p} = \left(\frac{u_{**}}{u_c} \right)^2 . \tag{4.17}$$

If $u_*/u \gg 1$, namely, very near the source branes, the turning point is

$$u_c = \left(\frac{u_*^{7-p}}{u_{**}^2} \right)^{1/(5-p)} . \tag{4.18}$$

In the non-extremal background, there may exist some points satisfying the turning-point condition $\dot{u} = 0$. From the effective potential (4.13) we find that the horizon, where $\tilde{f} = 0$, must be one of those points, regardless of the angular momentum. In particular, we notice that the central force exerted on the probe, defined as $F(u) = -dV(u)/du$, vanishes at the horizon. It means that once the probe reaches the horizon, it can stay at the horizon since the horizon is the turning point and the central force is zero there. In the previous section we have shown that the non-extremal branes are "located" at the horizon from the point of view of thermodynamics of a probe brane. Here we provide another evidence to support the argument from the dynamical aspect of a probe brane.

4.2 The D$(p-2)$-brane probe

In this subsection we consider the dynamics of a D$(p-2)$-brane in the background of (D$(p-2)$, Dp) bound states in the decoupling limit. In this case, the worldvolume is $(t, x_1, \ldots, x_{p-2})$, in the static gauge, one has the action of the probe

$$S_{p-2} = -\frac{T_{p-2}V_{p-2}}{g}\int d\tau H^{-1} \times \tag{4.19}$$
$$\times\left[(Hh^{-1})^{1/2}\sqrt{f-h(\dot{x}_{p-1}^2+\dot{x}_p^2)-H(f^{-1}\dot{r}^2+r^2\dot{\Omega}_{8-p}^2)}-(1-H_0)\sin\theta-H\right],$$

In the decoupling limit, it reduces to

$$S_{p-2} = -m_{p-2}\int d\tau\left(\frac{u}{R}\right)^{7-p}\times \tag{4.20}$$
$$\times\left[\sqrt{\frac{1}{(au)^{7-p}\tilde{h}}}\sqrt{\tilde{f}-\tilde{h}\left(\dot{\tilde{x}}_{p-1}^2+\dot{\tilde{x}}_p^2\right)-\left(\frac{R}{u}\right)^{7-p}\left(\tilde{f}^{-1}\dot{u}^2+u^2\dot{\Omega}_{8-p}^2\right)}-1+\frac{u_0^{7-p}}{2u^{7-p}}\right].$$

Here $m_{p-2} = V_{p-2}/[(2\pi)^{p-2}\tilde{g}]$ is the mass of the probe. When $au \gg 1$, this action approximates to

$$S_{p-2} = -m_{p-2}\int d\tau\left(\frac{u}{R}\right)^{7-p}\times \tag{4.21}$$
$$\times\left[\sqrt{\tilde{f}-\frac{1}{(au)^{7-p}}\left(\dot{\tilde{x}}_{p-1}^2+\dot{\tilde{x}}_p^2\right)-\left(\frac{R}{u}\right)^{7-p}\left(\tilde{f}^{-1}\dot{u}^2+u^2\dot{\Omega}_{8-p}^2\right)}-1+\frac{u_0^{7-p}}{2u^{7-p}}\right].$$

In fact, this is also the action of a D$(p-2)$-brane probe in the background produced by the source D$(p-2)$-branes (2.23). In the extremal limit, up to $\mathcal{O}(v^4)$, its motion is a geodesic of the following moduli space:

$$ds_m^2 = du^2 + u^2 d\Omega_{8-p}^2 + \frac{1}{\tilde{b}^2}\left(d\tilde{x}_{p-1}^2 + d\tilde{x}_p^2\right). \tag{4.22}$$

Therefore, in this approximation, the D$(p-2)$-brane probe moves as in a flat space. Namely, in the large noncommutative effect limit, the effect of the N_p coincident Dp-branes in the source on the motion of a D$(p-2)$-brane probe vanishes. The source looks like the one consisting of only D$(p-2)$-branes with two smeared coordinates and no B-field.

In the large $au \gg 1$ limit, if one does not consider the motion of the probe along the relative transverse directions (that is, setting $\dot{\tilde{x}}_{p-1} = \dot{\tilde{x}}_p = 0$ in the the action (4.21)), the action of the probe then reduces to that (4.2) of a noncommutative Dp-brane probe, except a difference in the mass of probes. If the mass of both probes is equal to each other, then they have the same interaction potential and the same phase shift depending on the distance and the velocity. Because we are considering only a single D-brane probe, to match the mass of probes, we should have $\delta N_p m_p = \delta N_{p-2} m_{p-2}$, from which one gets the relation (3.19) again. Thus, from the point of view of the dynamics of probes, we see again the equivalence between δN_p noncommutative Dp-branes and δN_{p-2} D$(p-2)$-branes with two smeared coordinates and no B-field.

Without the motion along the relative transverse directions, similarly to the case of the Dp-brane probe, we also obtain an effective central potential for the D$(p-2)$-brane probe as

$$V(u) = E\left\{1 - \frac{m_{p-2}\tilde{f}^2}{2E}\left(\frac{u}{R}\right)^{7-p}\left[1 - \frac{\mathcal{C}\tilde{f}}{\mathcal{B}^2}\right]\right\} + \frac{L^2 \tilde{f}^3}{2m_{p-2}u^2\mathcal{B}^2}\,, \qquad (4.23)$$

where we have not taken the large au limit and

$$\mathcal{C} = \frac{1+(au)^{7-p}}{(au)^{7-p}}\,, \qquad \mathcal{B} = 1 - \frac{u_0^{7-p}}{2u^{7-p}} + \frac{E}{m_{p-2}}\left(\frac{R}{u}\right)^{7-p}. \qquad (4.24)$$

Due to the appearance of $\mathcal{C}$, the motion of the probe D$(p-2)$-brane is a little different from that of the Dp-brane probe. However, we find that the horizon of the background is still the turning point of the probe and the central force on the D$(p-2)$-brane probe vanishes. This implies that the D$(p-2)$-brane probe can also stay at the horizon. It is also consistent with the result from the analysis of thermodynamics of the probe. Indeed, as an ingredient of the bound state (D$(p-2)$, Dp), non-extremal D$(p-2)$-branes should also be located at the horizon of the background.

5. Conclusions

As is well known by now, the worldvolume coordinates of Dp-branes will become noncommutative if a nonvanishing constant NS B-field is present on the worldvolume of the Dp-branes. The worldvolume theory is then the super Yang-Mills theory in a noncommutative space (noncommutative gauge field theory). Each of the nonthreshold (D$(p-2)$, Dp) bound states ($2 \le p \le 6$) can be viewed as a Dp-brane bound state with a nonvanishing NS B-field of rank two.

In this paper we have investigated two equivalent descriptions of the nonthreshold (D$(p-2)$, Dp) bound states in the dual gravity description. In the decoupling limit, the bound states can be described as Dp-branes with nonvanishing NS B-field, and then the worldvolume theory is a noncommutative gauge field with gauge group U(N_p) in $(p+1)$ dimensions (with two dimensions compactified on a torus in our case) if the number of the coincident Dp-branes is N_p. On the other hand, the nonthreshold (D$(p-2)$, Dp) bound state will reduce to the solution of D$(p-2)$-branes with two smeared coordinates and zero B-field in the decoupling limit and the large $au \gg 1$ limit. The latter condition is necessary for the validity of the dual gravity description. The worldvolume theory of the D$(p-2)$-branes should be an ordinary gauge field theory with gauge group U(N_{p-2}) in $(p+1)$ dimensions if the number of the coincident D$(p-2)$-branes is N_{p-2}. From the viewpoint of the thermodynamics of dual gravity solutions for the bound states (D$(p-2)$, Dp), we have found that N_p coincident Dp-branes with NS B-field is equivalent to N_{p-2} coincident D$(p-2)$-branes with two smeared coordinates and no B-field. In the equivalence, N_p and N_{p-2} must obey the relation (2.21), where $\tilde{V}_2$ is the area of the two additional dimensions and $\tilde{b}$ is a noncommutativity parameter. When the volume of the torus is sent to infinity keeping this relation, the ordinary super Yang-Mills theory reduces to the one with gauge group U(∞) in $(p-1)$ dimensions. We have identified the Yang-Mills coupling constant for the $(p-1)$-dimensional ordinary Yang-Mills theory.

We have also shown the equivalence from the thermodynamics and dynamics of two probes in the background of the bound states (D$(p-2)$, Dp). One of the probes is a bound state of Dp- and D$(p-2)$-branes, which we called a noncommutative Dp-brane probe. The other is a D$(p-2)$-brane probe. In the asymptotically flat limit, when the two probes fall into the horizon of the source from spatial infinity, their static interaction potentials at the horizon are converted into heat and thereby are absorbed by the source. In this process, the first law of black hole thermodynamics is obeyed. In the decoupling limit, we have found that the thermodynamics and dynamics of the two probes are identical if the numbers of probe branes satisfy the relation (3.19), completely the same relation as (2.21). As a byproduct, we have found that the free energy of the noncommutative Dp-brane probe in the Dp-brane background with a nonvanishing NS B-field is the same as that of a Dp-brane probe in the Dp-brane background without B-field. It shows that the thermodynamics of the noncommutative super Yang-Mills coincides with the ordinary case in the large-N limit, not only in the Higgs branch, but also in the Coulomb branch. In addition, from the analysis of dynamics of probes, we have derived that the non-extremal Dp-branes can be located at the horizon. Our discussions support the argument by Lu and Roy [11] that there is an equivalence between the noncommutative super Yang-Mills with gauge group U(N_p) in $(p+1)$ dimensions with two dimensions compactified on a torus and the ordinary one with gauge group U(N_{p-2}) in $(p-1)$ dimensions when the area of the torus $\tilde{V}_2 \to \infty$, with the relation (2.21) between N_p and N_{p-2}. This result is also consistent with the Morita equivalence [8].

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